Polynomial Approximations for the Transverse Magnetic Polarizabilities of Some Small Apertures

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Abstract—**Polynomial approximations are given for the magnetic polarizabilities of some small apertures of various shapes, as functions of the aperture width to length ratios, for the case where the applied magnetic field is across the narrower dimension of the aperture. The shapes considered are the rectangle, diamond, rounded end slot, and ellipse, of which only the last has an exact solution.**

I. Introduction

In an earlier paper [1], polynomial expressions were given for the electric polarizabilities of small apertures of the shapes shown in Fig. 1, namely the rectangle, diamond, rounded end slot, and ellipse. Although those expressions were not exact, all contained features which would exist in exact solutions if they could be found, and a comparison with previously published data indicated that the polynomials had sufficient accuracy to be used in many applications.

The question arises whether a similar approach might also provide useful results for the magnetic polarizabilities. An important difference between the electric and magnetic cases is that the magnetic polarizability for the tangential magnetic field along the major dimension of the aperture (here referred to as the longitudinal polarizability) is different from that with the tangential magnetic field transverse to the major dimension (here referred to as the transverse polarizability). There appears to be no general relationship between the two magnetic polarizabilities. Also, in the electric case, the polarizability is independent of the choice of reference directions, and that independence was used in [1] to relate the slope of the polarizability coefficient to its magnitude for some shapes at an aspect ratio of 1, thereby providing one equation for solution of the terms of the polynomial. There seems to be no equivalent result for magnetic polarizabilities.

This paper is concerned only with transverse magnetic polarizabilities, i.e., where the tangential magnetic field is transverse to the longer aperture dimension L, as shown in Fig. 1. Longitudinal polarizabilities do not appear to be

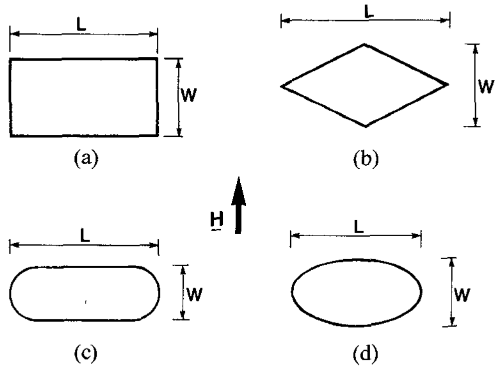

Fig. 1. Aperture shapes and direction of applied magnetic field. (a) Rectangle. (b) Diamond. (c) Rounded end slot. (d) Ellipse.

functions suitable for approximation by simple polynomials.

For each aperture shape, the transverse magnetic polarizability can be expressed in the form

$$P_m = f\left(\frac{W}{L}\right)L^3$$

in which $f(W/L)$ is a dimensionless coefficient. In all cases to follow, the ratio W/L will be designated α.

In the electric polarizability case, the behavior of the polarizability coefficient for each aperture shape was found to be quadratic for $\alpha \to 0$, and both the magnitude and slope were known for $\alpha = 1$. This provided enough information to define a fourth-power polynomial approximation. In the transverse magnetic case there is one less equation as there is no symmetry property, and from the behavior for $\alpha \to 0$ and the value at $\alpha = 1$ there are only enough equations to define a third-power polynomial approximation. Also, for the transverse magnetic polarizabilities there are fewer experimental data for comparison, as Cohn's electrolytic tank experiments [2] did not include that orientation. The need for more data for that situation has been recognized [3].

II. Behavior for $W \ll L$

Consider first the magnetic polarizabilities of the aperture shapes in Fig. 1 with $W \ll L$ and with the applied tangential magnetic field transverse to the aperture, i.e.,

Manuscript received April 8, 1986; revised July 26, 1986. This work was supported in part by the Australian Telecommunications and Electronics Research Board.

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IEEE Log Number 8611020.

0018-9480/87/0100-0020$01.00

parallel with the W dimension. In [4] Bethe gave the result for the magnetic polarizability P_m of a long slit (equivalent to Fig. 1(a) with $W \ll L$) and with transverse magnetic field as

$$P_m = \frac{\pi}{16} W^2 L$$

which may be interpreted as a polarizability of $(\pi/16)W^2$ per unit length. This suggests that if the width ω of a long narrow aperture varies very slowly along the length, then the transverse magnetic polarizability could be obtained by integrating $(\pi/16)\omega^2$ along the length of the aperture.

If applied to a long narrow ellipse, as in Fig. 1(d) but with $W \ll L$, this postulate gives for the polarizability

$$\frac{\pi}{24} W^2 L$$

which agrees with the exact solution from [5] (and that given in Section III of this paper) if that solution is applied to an ellipse of eccentricity approaching unity. The application of this reasoning to all of the shapes in Fig. 1 gives for $W \ll L$

$$P_m = \begin{cases} \frac{\pi}{16} W^2 L & \text{rectangle} \\ \frac{\pi}{48} W^2 L & \text{diamond} \\ \frac{\pi}{16} W^2 L & \text{rounded end slot} \\ \frac{\pi}{24} W^2 L & \text{ellipse.} \end{cases}$$

For each shape and for $0 \leqslant \alpha \leqslant 1$, the polarizability coefficient $f(\alpha)$ is approximated by a polynomial of the form

$$f(\alpha) = a + b\alpha + c\alpha^2 + d\alpha^3. \tag{1}$$

From the behavior for $W \ll L$, i.e., $\alpha \to 0$, then $a = 0$, $b = 0$, and c is known for each of the four aperture shapes. The remaining term is obtained from the value of $f(\alpha)$ at $\alpha = 1$. This gives a third-power polynomial, whereas in the electric polarizability case fourth-power polynomials were obtained. However, the criterion of usefulness is not the power of the polynomials but the extent to which the expressions adequately represent the properties of the functions. That is now to be determined.

III. Ellipse

The ellipse will be considered first as it is the only shape for which an analytical solution exists, and it can therefore be used as an indication of the accuracy of the method.

At $\alpha = 1$, the ellipse is a circle for which the magnetic polarizability is known [4], [5] to be $\frac{1}{6}L^3$. Thus, at $\alpha = 1$, $f(\alpha) = \frac{1}{6}$, so that the third-power polynomial approximation for the ellipse is

$$f(\alpha) = \frac{\pi}{24}\alpha^2\{1.0 + 0.2732\alpha\}. \tag{2}$$

TABLE I
Transverse Magnetic Polarizability Coefficient of an Ellipse as a Function of Aspect Ratio

α	Polynomial	Exact Value	Error
1.0	0.1667	0.1667	-
0.9	0.1321	0.1316	0.4%
0.8	0.1021	0.1013	0.8%
0.7	0.07641	0.07557	1.1%
0.6	0.05485	0.05404	1.5%
0.5	0.03720	0.03653	1.8%
0.4	0.02323	0.02275	2.1%
0.3	0.01275	0.01247	2.2%
0.2	0.005522	0.005406	2.1%
0.1	0.001345	0.001324	1.6%

From [4] and [5], the exact expression for the transverse magnetic polarizability coefficient for an ellipse, expressed in the notation of this paper, is

$$\frac{\pi\alpha^2(1-\alpha^2)}{24\left[E\left(\sqrt{1-\alpha^2}\right) - \alpha^2 K\left(\sqrt{1-\alpha^2}\right)\right]}$$

in which K and E are complete elliptic integrals of the first and second kinds, respectively, as defined in [6].

In Table I, values for the polarizability coefficient calculated from the polynomial are compared with those from the exact solution. The accuracy of the third-power polynomial approximation for the transverse magnetic polarizability is of the same order as that obtained with the fourth-power polynomial and electric polarizability [1].

IV. Rounded End Slot

The next shape for which a polynomial approximation for the polarizability coefficient will be given is the rounded end slot, as in Fig. 1(c). It also becomes a circle at $\alpha = 1$, and the polynomial is

$$f(\alpha) = \frac{\pi}{16}\alpha^2\{1.0 - 0.1512\alpha\}. \tag{3}$$

Probably the best information for comparison purposes is that which was used to prepare Fig. 5 of [7], in which the "dimensionless polarizability," defined as the aperture polarizability divided by $(\text{area})^{3/2}$, is plotted against aspect ratio for several aperture shapes. The authors of [7] have kindly supplied a copy of the report [8] containing the corresponding tables of numerical values. Note that the convention used in [7] and [8] resulted in the aperture polarizabilities being twice those of the convention of [4], [5], and this paper.

In Table II, values for the polarizability coefficient from the polynomial (3) are compared with numerical results interpreted from [8]. Agreement between the two sets of data in Table II is better than 1.8% for all values of α.

TABLE II
TRANSVERSE MAGNETIC POLARIZABILITY COEFFICIENT OF A ROUNDED END SLOT AS A FUNCTION OF ASPECT RATIO

α	Polynomial	Numerical Solution [8]
1.0	0.1666	0.1667
0.8	0.1105	0.1118
0.5	0.04538	0.04618
0.333	0.02072	0.02103
0.2	0.007616	0.007684
0.1	0.001934	0.001944

TABLE III
TRANSVERSE MAGNETIC POLARIZABILITY COEFFICIENT OF A RECTANGLE AS A FUNCTION OF ASPECT RATIO

α	Polynomial	Numerical Solution [8]
1.0	0.2596	0.2596
0.8	0.1580	0.1587
0.75	0.1371	0.1377
0.5	0.05699	0.05743
0.333	0.02416	0.02435
0.2	0.008360	0.008412
0.1	0.002027	0.002035

V. COMMENTS ON THE MAGNETIC POLARIZABILITY OF A SQUARE

The rectangle and the diamond both become squares for $\alpha = 1$, but there is no analytically determined magnetic polarizability for a square. Cohn [2] in his electrolytic tank analog experiments provided a value of $0.2590L^3$, where L is the side length of the square. McDonald, using the variational modal method described in [9], calculated $0.2496L^3$, and Fikhmanas and Fridberg [10] have published a lower bound of $0.251L^3$ and an upper bound of $0.280L^3$. More recently, using numerical methods, De Smedt and Van Bladel [7] calculated $0.2596L^3$, and Okon and Harrington [11] obtained $0.2581L^3$. The reason for the spread of approximately 4 percent in the calculated values for the magnetic polarizability of a square is not known at this time. For the transverse magnetic polarizability of a rectangle, of which the square is a special case, agreement between the results from [7] and [9] improves in percentage terms as the aspect ratio decreases, and is better than 1 percent for $\alpha = 0.1$. (For the electric polarizability of a square, the two numerical values referred to in [1] were $0.1116L^3$ and $0.1126L^3$, i.e., agreement within 1 percent. However, the Fikhmanas and Fridberg bounds [10] of $0.1131L^3$ and $0.1190L^3$ are not consistent with either of the numerical values.) For the purposes of illustrating the polynomial approximation method, the De Smedt and Van Bladel result of $0.2596L^3$ for the magnetic polarizability of a square will be used, and the values from the resulting polynomial expressions for the rectangle and diamond will be compared with De Smedt's calculated values [8] for those shapes. If a different result for the square is later found to be more accurate, the polynomials can be modified accordingly.

VI. RECTANGLE

The third-power polynomial which has the appropriate behavior for the rectangle as $\alpha \to 0$ and which has the value of 0.2596 at $\alpha = 1$ is

$$f(\alpha) = \frac{\pi}{16}\alpha^2\{1.0 + 0.3221\alpha\}. \qquad (4)$$

Values calculated from this expression are compared in Table III with those obtained from the dimensionless polarizabilities in [8]. Agreement is better than 0.8 percent for all values of α shown in the table.

TABLE IV
TRANSVERSE MAGNETIC POLARIZABILITY COEFFICIENT OF A DIAMOND AS A FUNCTION OF ASPECT RATIO

α	Polynomial	Numerical Solution [8]
1.0	0.09178	0.09178
0.8	0.05537	0.05468
0.75	0.04792	0.04709
0.5	0.01965	0.01908
0.333	0.008247	0.007964
0.2	0.002829	0.002734
0.1	0.0006808	0.0006602

VII. DIAMOND

At $\alpha = 1$ the diamond is a square, so that with L defined as in Fig. 1(c) and the polarizability coefficient for a square taken as 0.2596 from [8], then the polarizability coefficient for a diamond of $\alpha = 1$ is $0.2596/2\sqrt{2}$.

This leads to the following third-power polynomial approximation for the polarizability coefficient:

$$f(\alpha) = \frac{\pi}{48}\alpha^2\{1.0 + 0.4023\alpha\}. \qquad (5)$$

In Table IV, values calculated from (5) are compared with those derived from the numerical solution [8]. The maximum difference is approximately 3.5 percent, which is more than for the other three shapes considered, but still acceptable for many applications.

VIII. HIGHER ORDER POLYNOMIALS

An important feature of the polynomial approximation approach is that useful results can be obtained simply from consideration of properties which an exact solution must possess. That is particularly the case for the ellipse and the rounded end slot, as the third-power polynomial expressions for those shapes do not rely on any numerical solutions. For the rectangle and the diamond, a numerical value for the square is used.

Of course, if the polarizability of any shape is specified for α values in addition to $\alpha = 1$, higher order polynomials can be obtained. As an example, if the known polarizability value for the ellipse at $\alpha = 0.5$ is used, the following fourth-power polynomial approximation results:

$$f(\alpha)=\frac{\pi}{24}\alpha^2\{1.0+0.1919\alpha+0.0814\alpha^2\}.$$

Agreement with the exact solution is better than 0.9 percent, compared with 2.2 percent for the third-power polynomial.

Likewise, if numerical values at $\alpha=0.5$ from [8] are used to generate fourth-power expressions for the other shapes, the following polynomials result, with similar improvements in accuracy:

Rounded end slot $f(\alpha)=\frac{\pi}{16}\alpha^2\{1.0-0.0857\alpha-0.0654\alpha^2\}$

Rectangle $f(\alpha)=\frac{\pi}{16}\alpha^2\{1.0+0.3577\alpha-0.0356\alpha^2\}$

Diamond $f(\alpha)=\frac{\pi}{48}\alpha^2\{1.0+0.2620\alpha+0.1403\alpha^2\}.$

However, for most applications the third-power polynomials would probably be sufficient. There is also the possibility that the apparently increased accuracy of the higher order polynomials may be illusory, as the numerical values from which they are derived may not be sufficiently precise.

IX. Conclusions

Polynomial approximations have been given for the transverse magnetic polarizability coefficients of some small apertures. A comparison with other published data indicates that the approximations should have sufficient accuracy to be used in many applications. Multiplication of the coefficients by L^3 gives the polarizabilities of the respective apertures.

It has been assumed that the apertures are small in wavelengths and that the wall is of infinitesimal thickness. If either of these conditions is not satisfied, correction terms will be necessary.

Acknowledgment

The author is very grateful to Prof. J. Van Bladel and Dr. R. De Smedt of the University of Ghent for the supply of the numerical data from which the curves in [7] were prepared, and also to Dr. G. Fitz-Gerald of RMIT for consultations on mathematical aspects of the aperture coupling problem.

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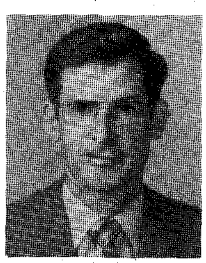

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